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# VARIABLE DENSITY EFFECTS IN COMBINED FREE AND FORCED CONVECTION IN INCLINED TUBES

M. IQBAL† and J. W. STACHIEWICZ‡

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## NOMENCLATURE

$A$, axial temperature gradient (assumed constant), $\partial T/\partial x$ [°F/ft];
$a$, tube radius [ft];
$h$, $= q/(T_w - T_b)$, heat-transfer coefficient for fully developed flow based on bulk temperature difference [Btu/sft² degF];
$L$, $= x/a$, axial distance [dimensionless];
$N_{Gr}$, $= \beta g A a^4/\nu_0^2$, Grashof number;
$N_{Ra}$, $= N_{Gr} N_{Pr}$, Rayleigh number;
$N_{Re}$, $= 2aU\rho/\mu$, Reynolds number;
$N_{Fr}$, $= U_0^2/ag$, Froude number;
$P^*$, $= p/\rho_0 U_0^2$, pressure [dimensionless];
$R$, $= r/a$, radius distance from the centre line of the tube [dimensionless];
$U_0, \rho_0$, average axial velocity and density at entrance to heat-transfer section;
$\beta$, volumetric coefficient of thermal expansion of the fluid [degF⁻¹];
$\nabla^2$, two-dimensional Laplacian in cylindrical coordinates;
$\nabla_1^2$, three-dimensional Laplacian in cylindrical coordinates;
$c_p, k, \mu, \rho$, fluid properties in the usual notation;
$\Phi$, $= -(T_w - T)/Aa$, difference between the wall temperature and any point of the fluid at the same section [dimensionless];
$\Phi_b$, $= -(T_w - T)/Aa$, difference between the wall temperature and bulk temperature of the fluid at the same section [dimensionless];
$\Phi_x, \Phi_r, \Phi_\theta$, $= v_x/U_0, v_r/U_0, v_\theta/U_0$, the axial radial and angular velocities respectively [dimensionless].

† Assistant Professor, Department of Mechanical Engineering, University of British Columbia, Vancouver.

‡ Professor, Department of Mechanical Engineering, McGill University, Montreal, Canada.

## INTRODUCTION

When, in non-isothermal flow, density differences which arise in the fluid due to temperature differences are suffiently large to produce large buoyancy forces under the action of the gravitational field, the resulting natural convection effects cannot be neglected and the buoyant force terms must be retained in the equations.

An indication of the relative magnitude of the forced and free convection effects may be obtained most readily by non-dimensionalizing the differential equations describing the flow, and by examining the relative magnitude of such parameters as the Reynolds, Grashof and Prandtl number. Of particular importance in the case of combined free and forced convection in closed conduits is the orientation of the gravitational field.

Theoretical studies of fully developed combined free and forced convection flow under uniform heat flux, inside straight vertical circular tubes, have been reported by Ostroumov [1], Hallman [2] and Morton [3] among others. Morton [4] studied the same for horizontal tubes. Iqbal and Stachiewicz [5] in a recent paper reported the study of the same phenomenon in inclined tubes and showed that for a given set of dimensionless parameters, there is a particular tube inclination that produces a maximum heat-transfer rate.

All the foregoing references considered density being variable only in the buoyancy terms of the momentum equations. For flow through vertical tubes, Koppel and Smith [6], Worsøe-Schmidt and Leppert [7] considered the effect of temperature dependence of density throughout the governing equations. Casal and Gill [8] investigated the same problem for horizontal tubes. This paper deals with inclined tubes.

## FORMULATION OF THE PROBLEM

Consider a straight circular tube of radius $a$ under uniform heat flux, and inclined at an angle $\alpha$ to the horizontal, within which a fully developed laminar flow is taking place. The buoyancy forces created within the fluid produce secondary flows distorting the normal Poiseuille flow to a form of helical motion. The thermophysical properties such as specific heat, thermal conductivity and viscosity are assumed constant. Density is considered linearly variable with temperature throughout the fluid, i.e. it is variable along the radial, angular as well as axial direction. Although the assumption of constant viscosity is unquestionably the least accurate, it is one that is made in most analytical studies such as Hallman's [2], Morton's [3, 4], etc.

Since the variation of density causes a change in axial velocity along the tube length, the axial pressure gradient $\partial p/\partial x$ is no longer constant. Flow is considered fully developed in the sense that the steep axial gradients of velocity and temperature near the tube entrance and at the beginning of the heated section have vanished and the velocity and thermal boundary layers have merged throughout the tube section. It is assumed that in this region, the axial gradients of the radial and the angular velocities are sufficiently small to be neglected. The axial gradient of the axial velocity is retained; it is, however, considered to be relatively small. In this region of "fully developed flow", the temperature difference $(T_w - T)$ between tube wall and the fluid at the same section is assumed independent of the distance along the tube length.

## ANALYSIS

Using the cylindrical polar coordinate system, (where $\theta$ is measured from the top vertical position of tube circumference) the governing equations for laminar flow under the foregoing assumptions can be written in dimensionless form as:

*Continuity equation.*

$$\frac{\partial}{\partial L}\left(\frac{\rho}{\rho_0}R\phi_x\right) + \frac{\partial}{\partial R}\left(\frac{\rho}{\rho_0}R\phi_r\right) + \frac{\partial}{\partial \theta}\left(\frac{\rho}{\rho_0}R\phi_\theta\right) = 0. \quad (1)$$

*Momentum equations.* In $L$, $R$, and $\theta$ directions these are respectively,

$$\frac{\rho}{\rho_0}\left\{\phi_x\frac{\partial \phi_x}{\partial L} + \phi_r\frac{\partial \phi_x}{\partial R} + \frac{\phi_\theta}{R}\frac{\partial \phi_x}{\partial \theta}\right\} = -\frac{\partial P^*}{\partial L} + \frac{2}{N_{Re}}\nabla_1^2\phi_x - \frac{ag\sin\alpha}{U_0^2}\cdot\frac{\rho}{\rho_0} \quad (2)$$

$$\frac{\rho}{\rho_0}\left\{\phi_r\frac{\partial \phi_r}{\partial R} + \frac{\phi_\theta}{R}\frac{\partial \phi_r}{\partial \theta} - \frac{\phi_\theta^2}{R}\right\} = -\frac{\partial P^*}{\partial R} + \frac{2}{N_{Re}}\left\{\nabla^2\phi_r - \frac{\phi_r}{R^2} - \frac{2}{R^2}\cdot\frac{\partial \phi_\theta}{\partial \theta}\right\} - \frac{ag\cos\theta\cos\alpha}{U_0^2}\cdot\frac{\rho}{\rho_0} \quad (3)$$

$$\frac{\rho}{\rho_0}\left\{\phi_r\frac{\partial \phi_\theta}{\partial R} + \frac{\phi_\theta}{R}\cdot\frac{\partial \phi_\theta}{\partial \theta} + \frac{\phi_r\phi_\theta}{R}\right\} = -\frac{1}{R}\frac{\partial P^*}{\partial \theta} + \frac{2}{N_{Re}}\left\{\nabla^2\phi_\theta + \frac{2}{R^2}\cdot\frac{\partial \phi_r}{\partial \theta} - \frac{\phi_\theta}{R^2}\right\} + \frac{ag\sin\theta\cos\alpha}{U_0^2}\cdot\frac{\rho}{\rho_0}. \quad (4)$$

*Energy equation.*

$$\frac{\rho}{\rho_0}\left\{\phi_r\frac{\partial \Phi}{\partial R} + \frac{\phi_\theta}{R}\cdot\frac{\partial \Phi}{\partial \theta} + \phi_x\right\} = \frac{2}{N_{Re}N_{Pr}}\cdot\nabla^2\Phi. \quad (5)$$

In this equation axial conduction and frictional dissipation are ignored.

*Equation of state.* Since the constant heat flux condition produces a linear variation of temperature with tube length (except in the entrance region), the density ratio $\rho/\rho_0$ becomes a linear function of distance, which in the dimensionless form can be written as,

$$\frac{\rho}{\rho_0} = 1 - N_{Ra}\frac{4N_{Fr}}{N_{Re}^2N_{Pr}}(L + \Phi). \quad (6)$$

It will be noticed in equation (6) that by varying the magnitude of the Froude number $N_{Fr}$, the ratio $\rho/\rho_0$ can be assigned any desired variation with temperature. Thus in this particular problem $N_{Fr}$ will be used to signify the variable density effect even though physically this has, of course, no meaning. For $N_{Fr} = 0$ and any specific value of $N_{Ra}$ other than zero the effect on the momentum equations will be the same as if the density was considered variable only in the buoyancy terms.

The equations (1–5) are to be solved under the following boundary conditions,

$$\phi_x(1,\theta) = \phi_r(1,\theta) = \phi_\theta(1,\theta) = \Phi(1,\theta) = 0 \quad (7)$$

$$\phi_x(0,\theta), \quad \phi_r(0,\theta), \quad \phi_\theta(0,\theta), \quad \Phi(0,\theta) \text{ are finite.} \quad (8)$$

In addition, the integral form of the continuity equation, given by the identity

$$\int_0^{2\pi}\int_0^1 \frac{\rho}{\rho_0}\phi_x R\,\mathrm{d}R\,\mathrm{d}\theta = \pi \quad (9)$$

must also be satisfied.

Equations (1–5) have been solved by a perturbation solution which expresses the dependent variables in power series of the Rayleigh number. The detailed method of solution is similar to that employed in [4, 5, 6]. As such, these details are not given here. Having obtained the velocity and temperature equations, the evaluation of the friction factor and the Nusselt number follows directly.

## FRICTION FACTOR

Fanning friction factor $f$, defined as: (shear stress per unit area on wetted wall)/(kinetic energy of the fluid per unit volume), can be expressed in dimensionless form as,

$$f = -\frac{4}{N_{Re}}\cdot\frac{1}{L}\int_0^L\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial \phi_x}{\partial R}\right]_{R=1}\mathrm{d}\theta\right\}\mathrm{d}L. \quad (10)$$

## NUSSELT NUMBER

The conventional definition of Nusselt number is

$$N_{Nu} = (2aq)/\{k(T_w - T_b)\} = -N_{Re}N_{Pr}/(2\Phi_b),$$

where $T_b$ is the bulk temperature of the fluid and $\Phi_b$ is the dimensionless bulk temperature difference.

## DISCUSSION OF RESULTS

*Velocity profiles*

The study of velocity profiles shows that for "constant density" case, i.e. $N_{Fr} = 0$, the profiles are similar to the ones given in [4], i.e. for horizontal case, the maximum velocity occurs below the tube centre line. As the tube inclination increases toward the vertical position, the position of maximum velocity shifts toward the centre, until for the vertical case the profiles become symmetrical about the centre. When the density is variable throughout the field, the axial velocity will be a function of axial length as shown for the horizontal case in Figs. 1 and 2. Comparison of Fig. 1 with 2 shows that for lower Prandtl number fluids (lower viscosity) the increase of axial velocity is larger than in the case of higher Prandtl number fluids.

*Friction factors*

The expression for the Fanning friction factor $f$, averaged over a length $L$ is given by equation (10). Friction factors have been plotted in Figs. 3–5. Figure 3 shows the variation of $f$ with Rayleigh number and with tube inclination. It appears that inclining the tube by about 30° above the horizontal has a considerably greater effect on friction factor than a variation in the tube inclination by the same amount from the vertical position.

Figure 4 shows the variation of friction factor with Reynolds numbers for various values of Rayleigh numbers. It shows that the buoyancy induced circulation has a considerable effect on the frictional characteristics of the flow. The effect on friction factor of introducing a variable density has also been investigated and a typical variation is shown in Fig. 5. This plot shows that variation in $N_{Fr}$

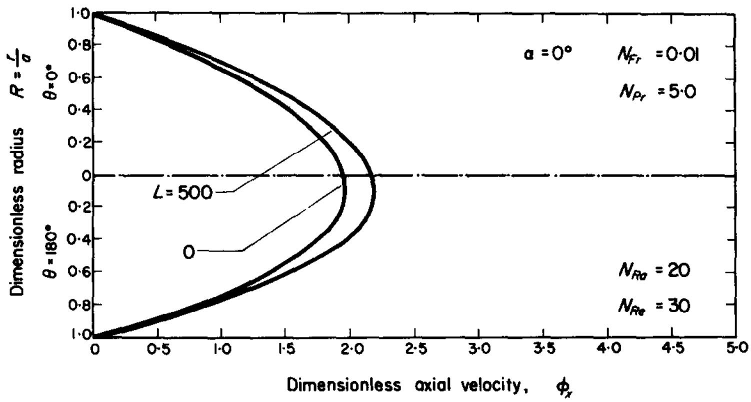


FIG. 1. Dimensionless axial velocity against dimensionless radius.

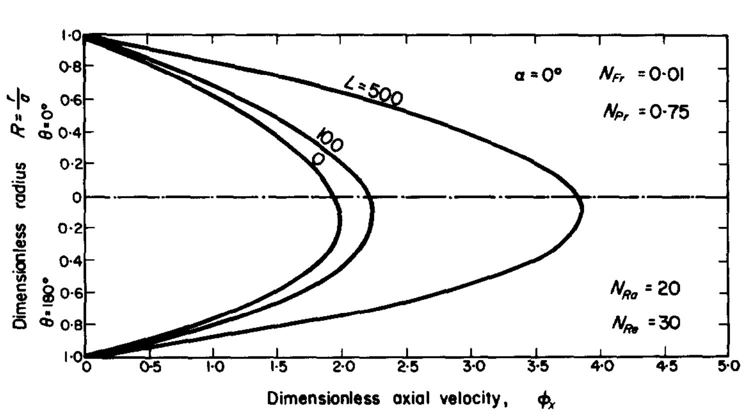


FIG. 2. Dimensionless axial velocity against dimensionless radius.

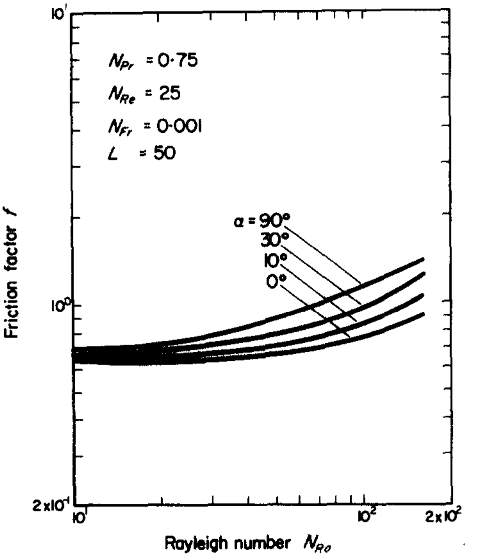


FIG. 3. Friction factor against Rayleigh number for various tube inclinations.

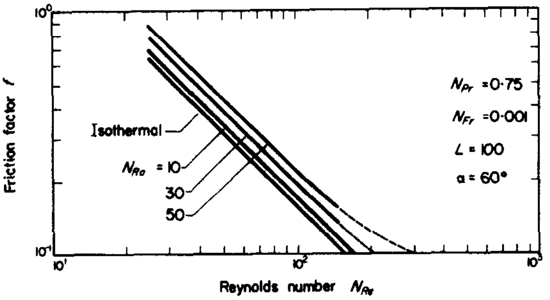


FIG. 4. Friction factor against Reynolds number for varying values of Rayleigh number.

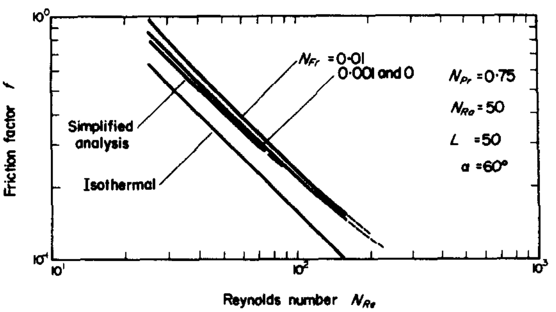


FIG. 5. Friction factor against Reynolds number for varying values of Froude number.

(i.e. in degree of dependence of density on temperature) affects the friction coefficient relatively little. The curve for $N_{Fr} = 0$ is almost identical to that for $N_{Fr} = 0{\cdot}001$ at Prandtl number of 0·75, (variation is even smaller at higher Prandtl numbers). There is, however, a difference of about 5–8 per cent between the values of $f$ obtained in the present analysis and those calculated by the simplified method in which density is assumed constant in all except the buoyancy terms of the momentum equations. This difference is shown in Fig. 5. A curve of isothermal friction factor is added in Fig. 5 to illustrate the effect of buoyant forces on friction factors.

*Temperature profiles and Nusselt numbers*

Study of temperature profiles and Nusselt numbers evaluated from the present analysis shows that the introduction of variable density has an insignificant effect. These plots remain the same as in [5]. It therefore appears that, as far as the heat-transfer analysis is concerned, the results would have been equally good if in the initial formulation of the continuity, momentum and energy equations, the variation of density with temperature was not taken into account, except of course, in the buoyancy terms of the momentum equations. This is a standard approach in free convection studies and the present results confirm its validity.

## CONCLUSIONS

Introduction of temperature dependent density function into all the equations governing flow in inclined tubes affects primarily the velocity field and only very slightly the temperature field.

The friction factor which, at a 60° tube inclination, is increased from 30–50 per cent over the isothermal value by the introduction of a buoyant force term into the momentum equations (while keeping the density constant in all other terms); is increased by a further 5–10 per cent when a more rigorous analysis is performed, in which the density is allowed to vary throughout the governing equations.

The Nusselt numbers on the other hand are essentially the same, at least within the range of Rayleigh numbers for which the perturbation analysis is valid, whether the simplified "constant density", or the variable density approach is used.

## ACKNOWLEDGEMENT

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# A CORRELATION OF PSYCHROMETRIC RATIOS FOR A FLAT PLATE

**J. Y. KAUH,† R. E. PECK and D. T. WASAN**
Department of Chemical Engineering, Illinois Institute of Technology, Chicago, Illinois

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## NOMENCLATURE

$C_p$, heat capacity;
$D$, molecular diffusivity;
$h_c$, convective heat-transfer coefficient;
$k_c$, heat conductivity;
$k_g$, convective mass-transfer coefficient;
$L$, length of a slab;
$M_m$, mean molecular weight;
$Nu_h$, Nusselt number for heat transfer, $h_cL/k_c$;
$Nu_m$, Sherwood number for mass transfer, $k_gL/D$;
$Pr$, Prandtl number, $C_p\mu/k_c$;
$Re$, Reynolds number, $u_sL/v$;
$Sc$, Schmidt number, $v/D$;
$St_h$, Stanton number for heat transfer, $Nu_h/Pr\,Re$;
$St_m$, Stanton number for mass transfer, $Nu_m/Sc\,Re$;
$u$, velocity in the axial direction;
$u_s$, free stream velocity;
$V$, velocity in the normal direction;
$\mu$, molecular viscosity;
$\rho$, density;
$B$, psychrometric ratio defined by equation (1);
$v$, kinematic viscosity;
$P_{bm}$, log mean inert partial pressure.

† J. Y. Kauh is presently employed with the Union Carbide Corporation, S. Charleston, West Virginia.

## INTRODUCTION

THE ANALOGY between momentum, heat and mass transfer has long been used for the correlation of heat- and mass-transfer coefficients for turbulent flow in pipes. The empirical extension of the methods used for pipes has been successfully employed for different shapes [1, 2, 3]. These correlations have steadily improved as more accurate knowledge of velocity and eddy viscosity distribution has been obtained.

The analogy between momentum, heat and mass transfer in turbulent flow has been used by several investigators including Bedingfield and Drew [4] and Lynch and Wilke [5, 6] for analyzing their psychrometric data. Even though much effort has been directed towards predicting the psychrometric ratio [7], no completely satisfactory agreement has been found among the numerous empirical relations which have been proposed. Wilke and Wasan [8] proposed a correlation for the psychrometric ratio based on their recent analysis of the transfer of heat and mass in pipe flow [9, 10].

The application of analogy for the prediction of the psychrometric ratio for a given geometry, should be possible if the necessary correlations of velocity distribution and friction coefficients for gas flow have been developed. Most of the existing psychrometric data have been obtained for a thermometer bulb of an approximately cylindrical shape. Wilke and Wasan made an attempt to apply the general form of the analogy for mass and heat transfer in pipes to